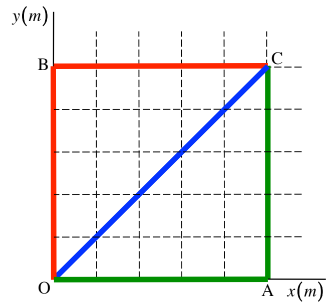
Problem 7.45

The force function is $\vec{F} = 2y\hat{i} + x^2\hat{j}$. Determine the work done by the force as one moves from:

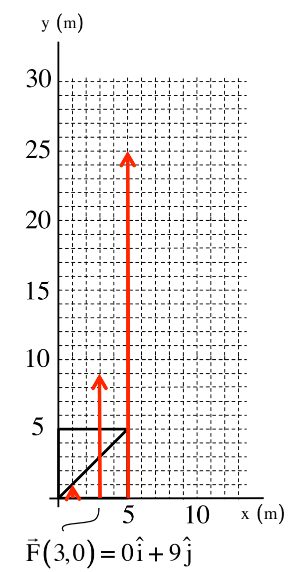
a.) 0-A-C (the green line):

This has the potential (no pun intended) of being a little more exciting because the force varies from point to point. To see this variation, the next two pages graph several force quantities at their respective points. There are a couple of additional points that have to be made before starting:



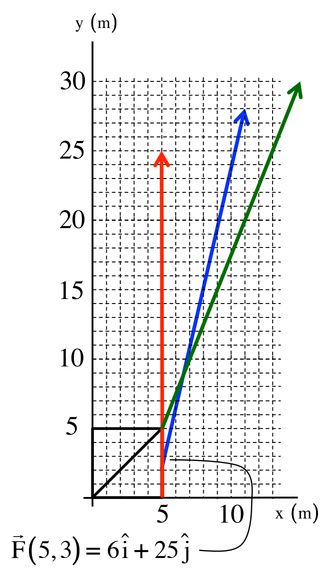
Differential work "dW" is done as a body moves over some differentially small path vector "$d\vec{r}$." In its most general form, "$d\vec{r} = (dx)\hat{i} + (dy)\hat{j}$," where "$(dx)\hat{i}$" is a differentially small displacement vector in the x-direction and "$(dy)\hat{j}$" is a differentially small displacement vector in the y-direction. Unfortunately, in this problem, the paths are such that certain of the variable are constant over certain parts of the path (along the horizontal axis, for instance, $dy = 0$ as y is constant), so we have to tailor each work integral to the path. You will see how that is done shortly. For now, look at the force graphs on the next several pages, be impressed with the amount of time it took me to hand-draw them, grid and all, then go on.

Along the bottom

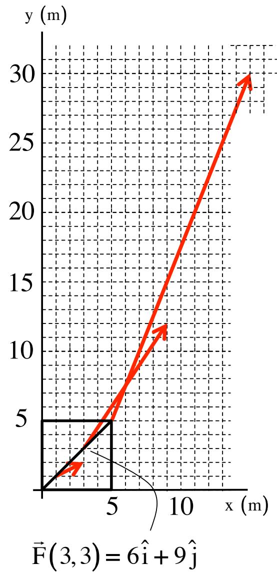


$\vec{F} = 2y\hat{i} + x^2\hat{j}.$

Along the right side

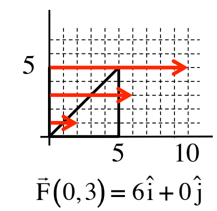


Along the diagonal

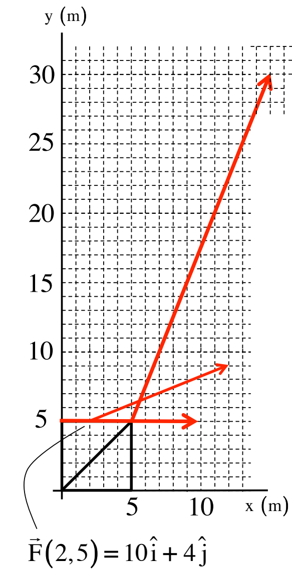


$\vec{F} = 2y\hat{i} + x^2\hat{j}.$

Along the left side

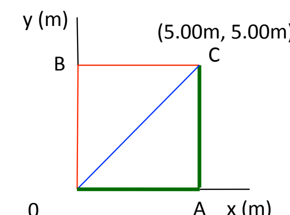


Along the top:



a.) 0-A-C (the green line):

To do this, we have to use two integrals, one along the horizontal section where the only displacement is "dx" as y = a constant along that path, and one for the vertical where the only displacement is "dy." I'll do both separately, then put them together:



Along O-A, y = 0 and dy = 0. That means $d\vec{r} = (dx)\hat{i} + (\overset{0}{\cancel{dy}})\hat{j}$

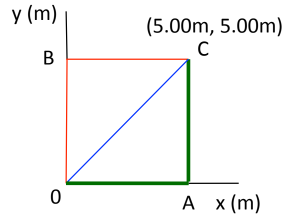
$$= (dx)\hat{i}$$

and

$$\begin{aligned} W &= \int dW_{O-A} \\ &= \int \vec{F} \bullet d\vec{r} \\ &= \int_{x=0}^{x=5} \left(2y\hat{i} + x^2\hat{j}\right) \bullet \left((dx)\hat{i}\right) \\ &= \int_{x=0}^{x=5} (2y)dx \\ &= (2yx)\Big|_{x=0}^{x=5} \\ &= 2(0)(5) - 2(0)(0) \\ &= 0 \text{ J} \end{aligned}$$

y = 0

Along A-C, x = 5 and dx = 0. That means



$$d\vec{r} = (\cancelto{0}{dx})\hat{i} + (dy)\hat{j}$$
$$= (dy)\hat{j}$$

and

$$W = \int dW_{A-C}$$
$$= \int \vec{F} \bullet d\vec{r}$$
$$= \int_{y=0}^{y=5} \left(2y\hat{i} + x^2\hat{j}\right) \bullet \left((dy)\hat{j}\right)$$
$$= \int_{y=0}^{y=5} \left(x^2\right)dy$$
$$= \left(x^2 y\right)\Big|_{y=0}^{y=5}$$
$$= (5)^2(5) - (5)^2(0)$$

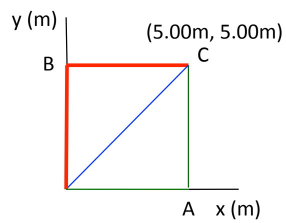
x = 5

$$= 125 \text{ J}$$

Apparently, the total work done along path O-A-C is 125 joules. (And yes, I got sloppy with the sig figs and units. Just don't do that on a test!)

b.) 0-B-C (the redline):

Along O-B, x = 0 and dx = 0. That means



$$d\vec{r} = (\cancelto{0}{dx})\hat{i} + (dy)\hat{j}$$
$$= (dy)\hat{j}$$

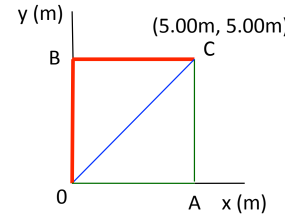
and

$$W = \int dW_{O-B}$$
$$= \int \vec{F} \bullet d\vec{r}$$
$$= \int_{y=0}^{y=5} \left(2y\hat{i} + x^2\hat{j}\right) \bullet \left((dy)\hat{j}\right)$$
$$= \int_{y=0}^{y=5} \left(x^2\right)dy$$
$$= \left(x^2 y\right)\Big|_{y=0}^{y=5}$$
$$= (0)^2(5) - (0)^2(0)$$

x = 0

$$= 0 \text{ J}$$

Along B-C, y = 5 and dy = 0. That means



$$d\vec{r} = (dx)\hat{i} + (\cancelto{0}{dy})\hat{j}$$
$$= (dx)\hat{i}$$

and

$$W = \int dW_{B-C}$$
$$= \int \vec{F} \bullet d\vec{r}$$
$$= \int_{x=0}^{x=5} \left(2y\hat{i} + x^2\hat{j}\right) \bullet \left((dx)\hat{i}\right)$$
$$= \int_{x=0}^{x=5} (2y)dx$$
$$= (2yx)\Big|_{x=0}^{x=5}$$
$$= 2(5)(5) - 2(5)(0)$$

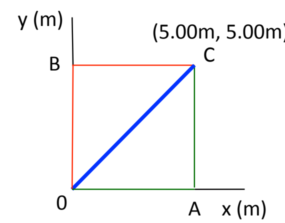
y = 5

$$= 50.0 \text{ J}$$

Apparently, the total work done along path O-A-C is 50.0 joules. (And the text book's solution is again wrong!)

c.) 0-C (the blue line):

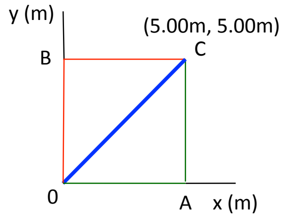
It's now time to use the full "dr" expression. I'm going to do it without limits to start, though. You will see why shortly:



$$W = \int dW_{O-C}$$
$$= \int \vec{F} \bullet d\vec{r}$$
$$= \int \left(2y\hat{i} + x^2\hat{j}\right) \bullet \left((dx)\hat{i} + (dy)\hat{j}\right)$$
$$= \int \left[(2y)dx + \left(x^2\right)dy\right]$$

At this point, it's time for a bit of trickery. Notice that along our path, x = y and dx = dy. Writing our integral completely in terms of "x," then, we get (see next page):

$$
\begin{aligned}
W &= \int \left[(2y)\,dx + \left(x^2\right)dy\right] \\
&= \int \left[(2x)\,dx + \left(x^2\right)dx\right] \\
&= \int_{x=0}^{x=5} \left(2x + x^2\right)dx \\
&= \left(x^2 + \frac{x^3}{3}\right)\Bigg|_{x=0}^{x=5} \\
&= \left((5)^2 + \frac{(5)^3}{3}\right) - 0 \\
&= 66.7\ \text{J}
\end{aligned}
$$



Apparently, the work the field does over path O-C is 66.7 joules.

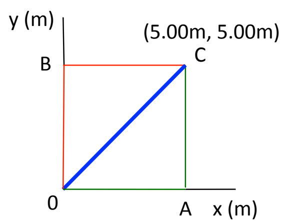
d.) Is the force field conservative?

No! In looking at the graphs on pages 2 and 3, clearly the forces along each path are distinct. That means the amount of work done around any closed path will dependent on the path, which means the field is non-conservative.

e.) Why are the work quantities different? See *Part d*.

SIDE NOTE:

Would it have been possible to have done *Part c* without the trickery? Let's try.



$$
\begin{aligned}
W &= \int dW_{O-C} \\
&= \int \vec{F} \bullet d\vec{r} \\
&= \int_{x=0,y=0}^{x=5,y=5} \left(2y\hat{i} + x^2\hat{j}\right) \bullet \left((dx)\hat{i} + (dy)\hat{j}\right) \\
&= \int_{x=0,y=0}^{x=5,y=0} \left[(2y)\,dx + \left(x^2\right)dy\right] \\
&= \left[(2yx) + \left(x^2y\right)\right]\Big|_{x=0,y=0}^{x=5,y=5} \\
&= \left[\left(2(5)(5) + (5)^2(5)\right) - (0)\right] \\
&= 175\ \text{J}
\end{aligned}
$$

Didn't work! So why not?

The problem is in the fact that y is a function of x (and dy a function of dx). If you don't take something like that into account in multi-dimensional Calculus, you end up with mush.